DIY: Logarithms: Solving logarithmic and exponential equations - A review

TO review rules of logarithms, simplifying expressions with logs, and solving logarithmic and exponential equations, watch the following set of YouTube videos explaining the basic techniques and rules, followed by some practice problems for you to try covering all the basic techniques, with answers and detailed solutions. Some additional resources are included for more practice at the end.

1. https://www.youtube.com/watch?v=Zw5t6BTQYRU (logarithm basics, natural logs, change of base)
2. https://www.youtube.com/watch?v=rBnQiLa2TYo (solving logarithmic equations)
3. https://www.youtube.com/watch?v=M6f6dANVyxA (solving exponential equations- examples)
4. https://www.youtube.com/watch?v=ls78_2UBcdY (Graphing Exponential Functions)
5. https://www.youtube.com/watch?v=jS_52xEjT_k (graphing logarithmic functions)
6. https://www.youtube.com/watch?v=GT6AYjgoFco (graphing logarithmic functions- another approach)

Properties of Logs: For any a > 0 and a ≠1:

1. $log_a x = b$ is equivalent to $a^b = x$ 2. $log_a x + log_a y = log_a xy$ 3. $log_a x - log_a y = log_a \frac{x}{y}$
4. $r * log_a x = log_a x^r$ $(log_a \frac{1}{x} = log_a x^{-1} = -log_a x)$ 5. $log_a 1 = 0$ 6. $log_a 0$ is undefined!
7. $\ln x = log_e x$ and $\log x = log_{10} x$ 8. Change of base formula: $log_a b = \frac{\log b}{\log a} = \frac{\ln b}{\ln a} = \frac{log_c b}{log_c a}$

Practice problems: The following problems use the techniques demonstrated in the above videos. The answers are given after the problems. Then detailed solutions, if you need them, are given after the answer section. For further assistance and help please contact ***Math Assistance Area***.

1. Rewrite in logarithmic form: a. $5^3 = 125$ b. $x^0 = 1$ $(x > 0, x \neq 1)$ c. $e^x = y$

2. Rewrite in exponential form: a. $log_3 \left(\frac{1}{9}\right) = -2$ b. $log_4 4096 = 6$ c. $\ln(10) \approx 2.3$

3. Solve the following: a. $log_4 64 =$ ____ b. $log_8 x = -\frac{1}{3}$ c. $log_x 256 = 4$

4. Using a calculator, find the following correct to three decimal places:

a. log 567 = _____ b. ln(4.75) = _________ c. log (-25) = _______ d. $\log(10^3)$ = _______

5. Combine the following into a single log term: $2log_3x + 3log_3y - log_3z =$____________

6. Expand the following expression to the sum/difference of individual logs:

$$ln\left[\frac{7a^4b^5}{4\sqrt{c}}\right]=$$

7. Solve the following exponential equations for x. Give your answers correct to 3 decimal places if needed.

a. $8^2 = 2^{3x+1}$ b. $\frac{1}{81} = 27^{4-x}$ c. $6^{2x} = 8$ d. $\frac{1}{4^x} = 20$

8. Solve the following logarithmic equations for x. Be sure to check your answer(s).

a. $log_3 15 = x$ (Hint: can be done by rewriting in exponential form then solving as in prob. 7.c.)

b. $log_2(x-1) + log_2(x+3) = log_2 63 - log_2 3$

c. $log_4 12x - log_4(x-3) = 2$

9. Application: The formula $A(t) = Pe^{rt}$ is used to calculate the amount in an account if *P* dollars are invested at an interest rate of *r* (as a decimal, so rate of 6% would have *r* = .06) for *t* years, compounded continuously. How long would it take for $100 invested in an account earning 4% interest, compounded continuously, to double in value (amount in the account would be $200)?

10. Sketch the graphs of the following on the same set of axes: $y = 2^x$ and $y = log_2 x$. (Make a table of values for each function. Hint: rewrite to logarithmic function in its equivalent exponential form).

Answers:

1. a. $log_5 125 = 3$ b. $log_x 1 = 0$ c. $\ln y = x$ $(or\ \ log_e y = x)$

2. a. $3^{-2} = \frac{1}{9}$ b. $4^6 = 4096$ c. $e^{2.3} \approx 10$

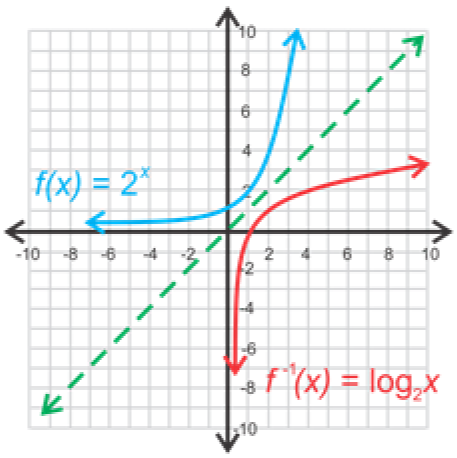
3. a. 3 b. x = ½ c. x = 4 4. a. 2.754 b. 1.558 c. not possible d. 3

5. $log_3\left(\frac{x^2y^3}{z}\right)$ 6. $\ln 7 + 4\ln a + 5\ln b - \ln 4 - \frac{1}{2}\ln c$ 7. a. x = 5/3 b. x = 16/3

7. c. x ≈ 0.580 d. x = -2.161 8. a. 2.465 b. x = 4 c. x = 12

9. 17.33 years (or about 7 years 4 months)

10.



(See the next page for detailed solutions to the above problems.)

Detailed Solutions

1\. a. $\underset{\text{base}}{5}^{\overset{\text{exponent}}{3}} = \underset{\text{result}}{125} \iff \log_{\underset{\text{base}}{5}} \overset{\text{result}}{125} = \underset{\text{exponent}}{3}$

b. $x^0 = 1 \iff \boxed{\log_x 1 = 0}$

c. $e^x = y \iff \log_e y = x$ but $\log_e = \ln$ so $\boxed{\ln y = x}$

2\. a. $\log_{\underset{\text{base}}{3}} \overset{\text{result}}{\left(\frac{1}{9}\right)} = \underset{\text{exponent}}{-2} \iff \boxed{3^{-2} = \frac{1}{9}}$

b. $\log_4 4096 = 6 \iff \boxed{4^6 = 4096}$

c. $\ln 10 \approx 2.3$ remember, $\ln = \log_e$ so $\log_e 10 \approx 2.3$

$$\boxed{e^{2.3} \approx 10}$$

the symbol "$\approx$" means "is approximately equal to".
The exact value of $\ln 10$ is an irrational number.

3\. a. $\log_4 64 = x \iff$ in exponential form, $4^x = 64$

$64 = 4(16) = 4 \cdot 4 \cdot 4 = 4^3$

since $4^x = 4^3$, $\boxed{x = 3}$

b. $\log_8 x = -\frac{1}{3} \iff 8^{-1/3} = x$ $\quad 8^{-\frac{1}{3}} = \frac{1}{8^{1/3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$

$\boxed{\frac{1}{2} = x}$

c. $\log_x 256 = 4 \iff x^4 = 256$

$$(x^4)^{1/4} = (256)^{1/4}$$

$$x = \sqrt[4]{256}$$

$$x = \sqrt[4]{2^8} = 2^{\frac{8}{4}} = 2^2 = 4$$

$256 = 2^8$

4. (no calculations – just use the $\boxed{\log}$ or $\boxed{\ln}$ keys of your calculator.)

5. $2\log_3 x + 3\log_3 y - \log_3 z$

using property 3 listed on page 1 $= \log_3 x^2 + \log_3 y^3 - \log_3 z$

using property 1, $= \log_3 (x^2 y^3) - \log_3 z$

using property 2, $= \boxed{\log_3 \left[\frac{x^2 y^3}{z}\right]}$

6. $\ln\left[\frac{7a^4 b^5}{4\sqrt{c}}\right] = \ln(7a^4b^5) - \ln(4c^{1/2})$

$\sqrt{c} = c^{1/2}$

$$= \ln 7 + \ln a^4 + \ln b^5 - (\ln 4 + \ln c^{1/2})$$

$$= \boxed{\ln 7 + 4\ln a + 5\ln b - \ln 4 - \tfrac{1}{2}\ln c}$$

7. a. $8^2 = 2^{3x+1}$ It is possible to write 8 as 2^3

$$(2^3)^2 = 2^{3x+1} \longrightarrow 2^6 = 2^{3x+1}$$

with equal bases, the exponents must be equal.

$$6 = 3x + 1$$
$$-1 \quad\quad -1$$
$$5 = 3x$$
$$\boxed{x = \frac{5}{3}}$$

7.b. $\frac{1}{81} = 27^{4-x}$ If possible, write each base as a power of the same number.

$(81)^{-1} = 27^{4-x}$ $81 = 9 \cdot 9 = 3 \cdot 3 \cdot 3 \cdot 3 = 3^4$ $27 = 3 \cdot 3 \cdot 3 = 3^3$

$(3^4)^{-1} = (3^3)^{4-x}$ or $3^{-4} = 3^{12-3x}$

so
$$\begin{aligned} -4 &= 12 - 3x \\ +3x & \quad\quad + 3x \\ \hline 3x - 4 &= 12 \\ +4 & \quad\quad +4 \\ \hline 3x &= 16 \end{aligned}$$
$$\boxed{x = \frac{16}{3}}$$

c. $6^{2x} = 8$ It is not possible to write both 6 and 8 as powers of the same base, so the technique for solving is to take the log or ln of both sides. (Remember: $\log = \log_{10}$)

Why only $\log_{10}$ and ln? Because these are the log bases available on most calculators.

$$\log 6^{2x} = \log 8$$
$$2x \cdot \log 6 = \log 8$$
$$x = \frac{\log 8}{2 \log 6} \approx \frac{.903089987}{2(.778151250)} \approx .580|2792109\ldots$$

round to 3 decimal places.

$$\approx \boxed{.580}$$

check: $6^{2(.58)} = 6^{1.16} = 7.9919995\ldots$
(close to 8)

d. $\frac{1}{4^x} = 20$ or $4^{-x} = 20$
$$\ln 4^{-x} = \ln 20$$
$$(-x) \ln 4 = \ln 20$$
$$x = \frac{\ln 20}{-\ln 4} \approx \frac{2.995732\ldots}{-1.386294\ldots} = -2.160|964\ldots$$
$$\approx \boxed{-2.161}$$

8.a. $\log_3 15 = x \iff 3^x = 15$ then $\log 3^x = \log 15$

$x \cdot \log 3 = \log 15$

$x = \dfrac{\log 15}{\log 3} \approx \dfrac{1.17609}{.477121}$

$= \boxed{2.465}$

Note: we could do this problem by directly applying the change of base formula (property 8 on page 1)

$\log_a b = \dfrac{\log b}{\log a}$ so $\log_3 15 = \dfrac{\log 15}{\log 3}$

we could have also said $\log_3 15 = \dfrac{\ln 15}{\ln 3} = \dfrac{2.70805}{1.09861} = 2.465$

b. $\log_2(x-1) + \log_2(x+3) = \log_2 63 - \log_2 3$

combining terms: $\log_2[(x-1)(x+3)] = \log_2\left(\frac{63}{3}\right)$

$\log_2(x^2+2x-3) = \log_2 21$

since the bases are the same, $x^2+2x-3 = 21$

$x^2+2x-24 = 0$

$(x+6)(x-4) = 0$

$x+6=0$ or $x-4=0$

$x=-6$ $\quad$ $x=4$

must check answers:

$x=-6$ $\quad \log_2(-6-1) + \log_2(-6+3)$ are undefined so $x=-6$ is not a possible solution.

$x=4$ $\quad \log_2(4-1) + \log_2(4+3) = \log_2 21$

$\log_2 3 + \log_2 7 = \log_2(3\cdot 7) = \log_2 21$ (checks)

so, only true solution is $\boxed{x=4}$

8.c. $\log_4 12x - \log_4 (x-3) = 2$

$$\log_4\left(\frac{12x}{x-3}\right) = 2$$

rewriting in exponential form: $4^2 = \frac{12x}{x-3}$

$$16 = \frac{12x}{x-3}$$

$$16(x-3) = 12x$$
$$16x - 48 = 12x$$
$$16x = 12x + 48$$
$$4x = 48 \longrightarrow \boxed{x = 12}$$

checking: $\log_4(12 \cdot 12) - \log_4(12-3) = 2$

$$\log_4 144 - \log_4 9 = 2$$

$$\log_4\left(\frac{144}{9}\right) = 2 \longrightarrow \log_4(16) = 2 \longrightarrow 4^2 = 16$$

(checks)

9. $A(t) = Pe^{rt}$ $\quad P = \$100 \quad r = .04 \quad A = 2P = 200$

(find t)

(note: the time it takes for an investment to double at 4% compounded continuously is the same regardless of the amount)

$$\frac{200}{100} = \frac{100e^{.04t}}{100} \longrightarrow e^{.04t} = 2$$

$$\ln e^{.04t} = \ln 2$$

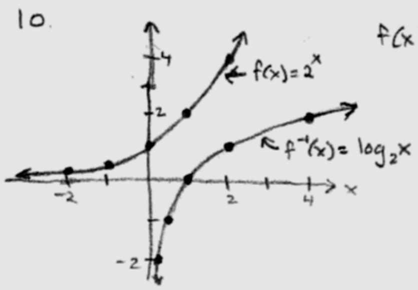
$$.04t \cdot \ln e = \ln 2 \qquad (\ln e = 1)$$

$$.04t = \ln 2$$

$$t = \frac{\ln 2}{.04} \approx \frac{.69314718}{.04} = 17.33 \text{ yrs.}$$

(or about 17 yrs 4 mo.)

10.



$f(x) = 2^x$

x	2^x
-2	1/4
-1	1/2
0	1
1	2
2	4

$f^{-1}(x) = \log_2 x$ (switch columns in $f(x)$ table)

x	$\log_2 x$
1/4	-2
1/2	-1
1	0
2	1
4	2

Want More Practice?

Click on the link below to download worksheets for more practice:

https://www.kutasoftware.com/freeia2.html Scroll down to the black entitled "Exponential and Logarithmic Functions" toward the bottom of the right-hand column. Click on any of the topics in that block to generate a worksheet on that topic. Answers to the problems are given at the end of the worksheet.