DIY: *Matrices: Introduction and Basic Operations*

To review the basics Matrices, what they represent, and how to find sum, scalar product, product, inverse, and determinant of matrices, watch the following set of YouTube videos. They are followed by several practice problems for you to try, covering all the basic concepts covered in the videos, with answers and detailed solutions. Some additional resources are included for more practice at the end.

1. Introduction to Matrices Introduction, matrix size, element notation

   Note: in the US, we usually denote a matrix **A** as [A], instead of the A with "~" underscore used in this presentation.
2. Matrix arithmetic basics some terminology, matrix addition, multiplication by a scalar, using equality of matrices
3. Matrix multiplication Matrix multiplication, properties, an application
4. Identity matrix

   Note: for an nxn identity matrix, the presenter uses the notation $\mathbf{I}_{nxn}$ but it is often denoted by $\mathbf{I}_n$.
5. Transpose of a matrix
6. Determinant of a matrix. Note: the determinant of matrix **A** can be denoted by |A| or det A.
   a. 2 x 2 matrix example
   b. Determinant of a 3 x 3 matrix determinant of a 3x3 matrix using cofactors
   c. Shortcut method This video demonstrates the method first mentioned but not demonstrated in the preceding video. Note: This method only works to find the determinant of a 3 x 3matrix.
   d. Using a calculator
7. Finding Inverses of matrices using:
   1. Finding a Matrix Inverse using Gauss-Jordan Elimination
   2. Matrix Inverse using Gauss-Jordan Elimination, Ex. 2 (3x3 matrix)
   3. Determinants and Cofactors for a 3x3 matrix

   Note: The method above works for any size matrix. For a 2x2 matrix, this method results in a familiar shortcut: To find the inverse, interchange the entries on the main diagonal (↘), change the signs on the entries on the other diagonal (↙), then divide each entry by the determinant of the matrix.

   Ex.
   $A=\begin{pmatrix} 2 & -5 \\ 8 & -15 \end{pmatrix}$ $|A| = \det(A) = (2)(-15)-(-5)(8) = 10$ $A^{-1} = \frac{1}{10}\begin{bmatrix} -15 & 5 \\ -8 & 2 \end{bmatrix} = \begin{bmatrix} -1.5 & 0.5 \\ -0.8 & 0.2 \end{bmatrix}$
8. Find solution to a system of equations using:
   a. Matrix Inverse: https://www.youtube.com/watch?v=AUqeb9Z3y3k&t=343s

      Note: a row or column matrix can also be referred to as a row or column "vector".
   b. Cramer's Rule: https://www.youtube.com/watch?v=BW6897HIOMA

Practice problems: The following problems use the techniques demonstrated in the above videos. The answers are given after the problems. Then detailed solutions, if you need them, are provided after the answer section. For further assistance and help please contact ***Math Assistance Area***.

1. Given **A** = $\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}$ and **B** = $[4 \quad -1 \quad 3]$

   a. the size of **A** is __________ b. the size of **B** is ___________

   c. **B** has only 1 _____ so can be called a _____ matrix. $\mathbf{B}^{\mathrm{T}}$ would be a ________ matrix.

   d. $\mathbf{A}^{\mathrm{T}} =$ _______ (Find the transpose of **A**)

   e. If **C** = **A**, then **C** = _____________.

   f. Solve for w, x, y, and z if: $\begin{bmatrix} x+1 & 3 \\ z & y-2 \end{bmatrix} = \begin{bmatrix} y & x-2 \\ w+4 & w \end{bmatrix}$

2. Find the **A** + **B** and **A** – **B** for the following matrices if possible

   a. **A**= $\begin{pmatrix} 2 & 4 & 6 \\ 1 & 0 & 1 \\ 0 & 2 & 9 \end{pmatrix}$; **B**= $\begin{pmatrix} 1 & 0 & 2 \\ 11 & -6 & 4 \\ 0 & 22 & 10 \end{pmatrix}$

   b. **A**= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; **B** = $\begin{pmatrix} 11 & -2 \\ 1 & 0 \\ 254 & 6 \end{pmatrix}$

   c. Using the matrices in part a, what is 3 **A** – 2 **B** ?

3. a. Write a 5 x 5 identity matrix. ($\mathbf{I}_5$)

   b. If **A** = $\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}$, then **IA** = _______ where the size of **I** is ________.

   Also, **AI** = _____ where the size of **I** is ___________.

   c. Which is true? $\mathbf{0}_{2,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. If **0** is the zero matrix, which of the following is (are) true?

1. **0A** = **A**
2. **0A = 0**
3. **0** + **A** = **A**
4. **0** + **A** = **0**
5. a zero matrix is always a square matrix

4. Find **AB** and **BA** if possible

a. $\mathbf{A} = \begin{pmatrix} 4 & 2 & -1 \\ 7 & 0 & 10 \\ 1 & -3 & 0 \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$

b. $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & -7 \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

c. $\mathbf{A} = \begin{pmatrix} 7 & 0 \\ 1 & -7 \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} 13 & 2 \\ 1 & 5 \\ 0 & 1 \\ 0 & 24 \end{pmatrix}$

d. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

5. Find |A| for the following matrices, if possible:

a) **A**= $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) **A**= $\begin{pmatrix} 2 & 6 \\ 3 & 0 \end{pmatrix}$

c) **A** = $\begin{pmatrix} -2 & 4 & -9 \\ 1 & -5 & -6 \\ -3 & -1 & -7 \end{pmatrix}$

d) **A** = $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & -2 & -1 & -5 \\ -1 & 0 & -8 & -9 \\ 3 & -1 & -10 & 11 \end{pmatrix}$

e) **A** = $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 14 & 0 \end{bmatrix}$

6. Find A^{-1} if it exists.

a) **A** = $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

b) **A** = $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

c) **A** = $\begin{bmatrix} 2 & -1 \\ 1 & -2 \\ 4 & 3 \end{bmatrix}$

d) **A** = $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 14 & 0 \end{bmatrix}$

e) **A** = [2]

7. Determine whether the following matrices are inverses of each other

a) $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

8. Find the solution of the following system of equations if it exists using matrices.

Using the inverse of the coefficient matrix

a) $\frac{1}{2}x + \frac{1}{3}y = \frac{49}{18}$
$\frac{1}{2}x + 2y = \frac{4}{3}$

Using the inverse of the coefficient matrix

b) $3x + 2y = 6$
$y - 6x = -27$

Using Cramer's Rule

c) 2x - 3y = -7
–x - 9y = 3

Using Cramer's rule

d) 3x - 5y = 2
5x - 6z = 22
-5y – z = -3

Answers:

1. a. 3 x 2 b. 1 x 3 c. row; row; column d. $\begin{bmatrix} 2 & 4 & 0 \\ 3 & 1 & 7 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}$ f. { x, y, z, w } = { 5, 3, 8, 4}

2. a. **A** + **B** = $\begin{bmatrix} 3 & 4 & 8 \\ 12 & -6 & 5 \\ 0 & 24 & 19 \end{bmatrix}$ **A** – **B** = $\begin{bmatrix} 1 & 4 & 4 \\ -10 & 6 & -3 \\ 0 & -20 & -1 \end{bmatrix}$

b. neither **A** + **B** nor **A** – **B** is possible c. $\begin{bmatrix} 4 & 12 & 14 \\ -19 & 12 & -5 \\ 0 & -38 & 7 \end{bmatrix}$

3. a. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b. **A** , 3 x 3 , **A** , 2 x 2 c. **0** = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ d. only 2 and 3 are true

4. a. $\begin{bmatrix} 9 \\ -43 \\ 1 \end{bmatrix}$, **BA** is not possible b. $\begin{bmatrix} 6 & 8 & 2 \\ 3 & -5 & 1 \end{bmatrix}$, **BA** is not possible

c. **AB** is not possbile, **BA** = $\begin{bmatrix} 93 & -14 \\ 12 & -35 \\ 1 & -7 \\ 24 & -168 \end{bmatrix}$

d. **AB** = $\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, **BA** = $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}$ (Note that **AB** and **BA** are *reflections* of the original **B**.

5. a) det **A** does not exist. The determinant is only defined for *square* matrices.

b) det **A** = -18 c) det **A** = 186 d) |**A**| = 336 e) det **A** = 0

6. a) $\mathbf{A^{-1}} = \begin{bmatrix} 1/6 & 2/3 \\ 1/6 & -1/3 \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{-1}{3} \end{bmatrix}$ c) not a square matrix. No inverse.

d) since det **A** = 0, **A** is a singular matrix. It has no inverse.

Note: if **A** were the coefficient matrix for a system of 3 equations, the system would have either no solution or an infinite number of solutions.

e) det [2] = [½]

7. a) yes, they are inverses b) no, they are not inverses

8. a) (x, y) = (6, -5/6) b) (x, y) = (4, -3) c) (x, y) = $\left(\frac{-24}{7}, \frac{1}{21}\right)$ d) (x, y, z) = $(3, -2, 4)$

Detailed Solutions

1.a. [A] has 3 rows and 2 columns, so is $\boxed{3\times 2}$

b. [B] has 1 row and 3 columns, so is 1×3

c. [B] has only 1 <u>row</u> so can be called a row <u>matrix</u>. $[B]^T$ would be a <u>column</u> matrix

$$\text{if } [B]=\begin{bmatrix}4 & -1 & 3\end{bmatrix} \text{ then } [B]^T=\begin{bmatrix}4\\-1\\3\end{bmatrix}$$

d. $[A]^T = \begin{bmatrix}2 & 4 & 0\\ 3 & 1 & 7\end{bmatrix}$ ← column 1 becomes row 1 in transpose; ← column 2 in [A] becomes row 2 in $[A]^T$

e. If $[C]=[A]$, then $[C]=\begin{bmatrix}2 & 3\\ 4 & 1\\ 0 & 7\end{bmatrix}$ All entries must be equal to the corresponding entries in [A].

f. $\begin{bmatrix}x+1 & 3\\ z & y-2\end{bmatrix}=\begin{bmatrix}y & x-2\\ w+4 & w\end{bmatrix}$ Equating corresponding entries:

$x+1=y \quad 3=x-2 \quad z=w+4 \quad y-2=w$

Solving $3=x-2 \rightarrow \boxed{x=5}$

using $x=5$, solve $y=x+1$ $\boxed{y=6}$

using $y=6$, solve $y-2=w$ $\boxed{w=4}$

or $\boxed{(x,y,z,w)=(5,6,8,4)}$

then, using $w=4$, solve $z=w+4$ $\boxed{z=8}$

2. a. $[A]+[B]=\begin{bmatrix}2 & 4 & 6\\ 1 & 0 & 1\\ 0 & 2 & 9\end{bmatrix}+\begin{bmatrix}1 & 0 & 2\\ 11 & -6 & 4\\ 0 & 22 & 10\end{bmatrix}=\begin{bmatrix}2+1 & 4+0 & 6+2\\ 1+11 & 0+(-6) & 1+4\\ 0+0 & 2+22 & 9+10\end{bmatrix}=\boxed{\begin{bmatrix}3 & 4 & 8\\ 12 & -6 & 5\\ 0 & 24 & 19\end{bmatrix}}$

2. (cont.)

$$[A]-[B] = \begin{bmatrix} 2-1 & 4-0 & 6-2 \\ 1-11 & 0-(-6) & 1-4 \\ 0-0 & 2-22 & 9-10 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 4 & 4 \\ -10 & 6 & -3 \\ 0 & -20 & -1 \end{bmatrix}}$$

b. [A] and [B] cannot be added or subtracted because they are not the same size.

c. $$3[A]-2[B] = 3\begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 1 \\ 0 & 2 & 9 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 11 & -6 & 4 \\ 0 & 22 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 & 18 \\ 3 & 0 & 3 \\ 0 & 6 & 27 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 4 \\ 22 & -12 & 8 \\ 0 & 44 & 20 \end{bmatrix} = \begin{bmatrix} 6-2 & 12-0 & 18-4 \\ 3-22 & 0-(-12) & 3-8 \\ 0-0 & 6-44 & 27-20 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & 12 & 14 \\ -19 & 12 & -5 \\ 0 & -38 & 7 \end{bmatrix}}$$

3. a. I_5 has "1" for each diagonal entry ($i_{11}, i_{22}, i_{33}, i_{44}, i_{55}$) and "0" for all other entries.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b. $$[IA] = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{3\times 2}{\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}} = \begin{bmatrix} \overbrace{1(2)+0(4)+0(0)}^{\text{(row 1 of I)(column 1 of A)}} & \overbrace{1(3)+0(1)+0(7)}^{\text{(row 1 of I)(column 2 of A)}} \\ 0(2)+1(4)+0(0) & 0(3)+1(1)+0(7) \\ 0(2)+0(4)+1(0) & \underbrace{0(3)+0(1)+1(7)}_{\text{(row 3 of I)(column 2 of A)}} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 7 \end{bmatrix}$$

equal

$= [A]$. $[I]$ is 3×3

Also, $[A][I] = [A]$ but here, $[I]$ is 2×2 $\quad \underset{3\times 2}{[A]}\cdot\underset{2\times 2}{[I]}$ must be equal

3. (cont) c. $O_{2,2}$ is a 2×2 zero matrix which has all entries = 0

$$O_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

d.
1. $OA = A$ False $\quad O \cdot A = O$
2. $OA = O$ True
3. $O + A = A$ True $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad O + A = \begin{bmatrix} 0+a & 0+b \\ 0+c & 0+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
4. $O + A = O$ False
5. "a zero matrix is always a square matrix" is false

for example, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}$ then for $A + O$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

The zero matrix assumes whatever size is required for the operation in which it is used.

4. a. $\underset{3\times 3}{[A]} \cdot \underset{3\times 1}{[B]} = \begin{bmatrix} 4 & 2 & -1 \\ 7 & 0 & 10 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 4(1)+2(0)+(-1)(-5) \\ 7(1)+0(0)+10(-5) \\ 1(1)+(-3)(0)+0(-5) \end{bmatrix} = \boxed{\begin{bmatrix} 9 \\ -43 \\ 1 \end{bmatrix}}$

same

size of product

$\underset{3\times 1}{[B]} \cdot \underset{3\times 3}{[A]}$

these dimensions are not the same, So BA is not possible.

b. $\underset{2\times 2}{[A]} \cdot \underset{2\times 3}{[B]} = \begin{bmatrix} 2 & 4 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2(3)+4(0) & 2(2)+4(1) & 2(1)+4(0) \\ 1(3)+(-7)(0) & 1(2)+(-7)(1) & 1(1)+(-7)(0) \end{bmatrix}$

equal

size of product

$$= \begin{bmatrix} 6+0 & 4+4 & 2+0 \\ 3+0 & 2-7 & 1+0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 8 & 2 \\ 3 & -5 & 1 \end{bmatrix}}$$

$\underset{2\times 3}{[B]} \cdot \underset{2\times 2}{[A]}$

not equal, so B·A is not possible

4. cont. c. $[A][B]$

$\underset{2\times 2}{}\ \underset{4\times 2}{}$ not equal. $[A][B]$ is not possible.

$$[B][A] = \begin{bmatrix} 13 & 2 \\ 1 & 5 \\ 0 & 1 \\ 0 & 24 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} 13(7)+2(1) & 13(0)+2(-7) \\ 1(7)+5(1) & 1(0)+5(-7) \\ 0(7)+1(1) & 0(0)+1(-7) \\ 0(7)+24(1) & 0(0)+24(-7) \end{bmatrix}$$

4×2 2×2 equal; size of product

$$= \begin{bmatrix} 91+2 & 0-14 \\ 7+5 & 0-35 \\ 0+1 & 0-7 \\ 0+24 & 0-168 \end{bmatrix} = \boxed{\begin{bmatrix} 93 & -14 \\ 12 & -35 \\ 1 & -7 \\ 24 & -168 \end{bmatrix}}$$

d. $$[A][B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0(1)+0(4)+1(7) & 0(2)+0(5)+1(8) & 0(3)+0(6)+1(9) \\ 0(1)+1(4)+0(7) & 0(2)+1(5)+0(8) & 0(3)+1(6)+0(9) \\ 1(1)+0(4)+0(7) & 1(2)+0(5)+0(8) & 1(3)+0(6)+0(9) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+7 & 0+0+8 & 0+0+9 \\ 0+4+0 & 0+5+0 & 0+6+0 \\ 1+0+0 & 2+0+0 & 3+0+0 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}}$$

$$[B][A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1(0)+2(0)+3(1) & 1(0)+2(1)+3(0) & 1(1)+2(0)+3(0) \\ 4(0)+5(0)+6(1) & 4(0)+1(5)+0(6) & 4(1)+5(0)+6(0) \\ 7(0)+8(0)+9(1) & 7(0)+8(1)+9(0) & 7(1)+8(0)+9(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+3 & 0+2+0 & 1+0+0 \\ 0+0+6 & 0+5+0 & 4+0+0 \\ 0+0+9 & 0+8+0 & 7+0+0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}}$$

Note: Although $[A]$ looks similar to an identity matrix, instead of leaving the other matrix unchanged, it produces a reflection of the other matrix.

5\. a) det A doesn't exist since A is not a square matrix.

b) $\det\begin{bmatrix} 2 & 6 \\ 3 & 0 \end{bmatrix} = 2(0) - (6)(3) = 0 - 18 = \boxed{-18}$

c) $\det\begin{bmatrix} -2 & 4 & -9 \\ 1 & -5 & -6 \\ -3 & -1 & -7 \end{bmatrix} = +(-2)\begin{vmatrix} -5 & -6 \\ -1 & -7 \end{vmatrix} - (4)\begin{vmatrix} 1 & -6 \\ -3 & -7 \end{vmatrix} + (-9)\begin{vmatrix} 1 & -5 \\ -3 & -1 \end{vmatrix}$

$= -2\left[(-5)(-7) - (-6)(-1)\right] - 4\left[1(-7) - (-6)(-3)\right]$

$-9\left[1(-1) - (-5)(-3)\right]$

$= -2(35 - 6) - 4(-7 - 18) - 9(-1 - 15)$

$= -2(29) - 4(-25) - 9(-16)$

$= -58 + 100 + 144 = \boxed{186}$

** d. The +/- sign for each cofactor of the determinant is determined by its location. For an element a_{ij}, if $i+j$ = odd, the sign is (−)
if $i+j$ = even, the sign is (+)

For a 4x4 matrix, the signs are $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$ ex. a_{12}: 1+2 = odd, so sign is (−)

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & -2 & -1 & -5 \\ -1 & 0 & -8 & -9 \\ 3 & -1 & -10 & 11 \end{vmatrix} = +(1)\begin{vmatrix} -2 & -1 & -5 \\ 0 & -8 & -9 \\ -1 & -10 & 11 \end{vmatrix} - (0)\begin{vmatrix} 2 & -1 & -5 \\ -1 & -8 & -9 \\ 3 & -10 & 11 \end{vmatrix} + (0)\begin{vmatrix} 2 & -2 & -5 \\ -1 & 0 & -9 \\ 3 & -1 & 11 \end{vmatrix}^{*}$$

$$-(1)\begin{vmatrix} 2 & -2 & -1 \\ -1 & 0 & -8 \\ 3 & -1 & -10 \end{vmatrix}$$

*terminology: in this term, $+0\begin{vmatrix} & \end{vmatrix}$ is called cofactor$_{13}$ or C_{13}

The "0" is entry$_{13}$. After eliminating row 1 and column 3, the determinant of the remaining 3x3 matrix is called "minor$_{13}$" or M_{13}.

Note:
The second and third terms are $0\cdot\begin{vmatrix} & \end{vmatrix} = 0$, so these determinants do not need to be calculated.

$$= 1\left[-2\begin{vmatrix}-8 & -9\\-10 & 11\end{vmatrix} - (-1)\begin{vmatrix}0 & -9\\-1 & 11\end{vmatrix} + (-5)\begin{vmatrix}0 & -8\\-1 & -10\end{vmatrix}\right] - 0 + 0$$

$$-1\left[+2\begin{vmatrix}0 & -8\\-1 & -10\end{vmatrix} - (-2)\begin{vmatrix}-1 & -8\\3 & -10\end{vmatrix} + (-1)\begin{vmatrix}-1 & 0\\3 & -1\end{vmatrix}\right]$$

$$= -2(-88-90) + (0-9) - 5(0-8) - 1\left[2(0-8) + 2(10+24) - 1(1-0)\right]$$

$$= -2(-178) - 9 + 40 - \left[-16 + 68 - 1\right] = 356 - 9 + 40 + 16 - 68 + 1$$

$$= 356 + 40 + 16 + 1 - 9 - 68$$

$$= 413 - 77$$

$$= \boxed{336}$$

** Finding the determinant of anything larger than a 3×3 matrix is tedious and highly subject to arithmetic errors. These are usually done on a graphing calculator or computer.

e. $\det A = \begin{vmatrix}2 & 1 & 3\\-1 & 4 & -2\\1 & 14 & 0\end{vmatrix}$ Using the shortcut method — copy left and middle columns to the right of the right column. Then add the products of the forward diagonals and subtract the products of the backwards diagonals

$$= \begin{array}{ccccc} \oplus & \oplus & \oplus\ominus & \ominus & \ominus \\ 2 & 1 & 3 & 2 & 1 \\ -1 & 4 & -2 & -1 & 4 \\ 1 & 14 & 0 & 1 & 14 \end{array}$$

$$= 2(4)(0) + (1)(-2)(1) + (3)(-1)(14)$$
$$- (3)(4)(1) - (1)(-1)(0) - (2)(-2)(14)$$

$$= 0 - 2 - 42 - 12 - 0 + 56 = -56 + 56 = \boxed{0}$$

6. a)

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\overbrace{\quad}^{A}} & \multicolumn{2}{c}{\overbrace{\quad}^{I}} \\ 2 & 4 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_1]{R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 6 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2/6 \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & \frac{1}{6} & -\frac{2}{6} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & \frac{1}{6} & -\frac{1}{3} \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{2}{3} \\ 0 & 1 & \frac{1}{6} & -\frac{1}{3} \\ \multicolumn{2}{c|}{\underbrace{\quad}_{I}} & \multicolumn{2}{c}{\underbrace{\quad}_{A^{-1}}} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/6 & 2/3 \\ 1/6 & -1/3 \end{bmatrix}$$

b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 3R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/(-2) \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3/(-3) \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right] \xrightarrow{R_2 - \frac{3}{2}R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

c) [A] has no inverse since it is not a square matrix.

d) [A] is the same matrix from 5.e. so we know that det $A = 0$. Therefore [A] has no inverse. But here is what happens if we try to find $[A]^{-1}$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 4 & -2 & 0 & 1 & 0 \\ 1 & 14 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_1]{R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 14 & 0 & 0 & 0 & 1 \\ -1 & 4 & -2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 14 & 0 & 0 & 0 & 1 \\ 0 & 18 & -2 & 0 & 1 & 1 \\ 0 & -27 & 3 & 1 & 0 & -2 \end{array}\right] \xrightarrow[R_3/(-27) \to R_3]{R_2/18 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 14 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/9 & 0 & 1/18 & 1/18 \\ 0 & 1 & -1/9 & -1/27 & 0 & 2/27 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 1 & 14 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/9 & 0 & 1/18 & 1/18 \\ 0 & 1 & -1/9 & -1/27 & 0 & 2/27 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 14 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/9 & 0 & 1/18 & 1/18 \\ \underbrace{0 \quad 0 \quad 0} & & & -1/27 & -1/18 & 1/54 \end{array}\right]$$

With a bottom row of all zeros, we cannot continue

e. if $[A] = [2]$, then $\boxed{[A]^{-1} = [1/2]}$ $\quad [2 \mid 1] \xrightarrow{\frac{1}{2}R_1} [1 \mid \frac{1}{2}]$

$[A][A]^{-1} = [2][1/2] = [1]$ and $[A]^{-1}[A] = [1/2][2] = [1]$

7. a. $$\underset{[A]}{\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}} \cdot \underset{[B]}{\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}} = \begin{bmatrix} 5(3) + 7(-2) & 5(-7) + (7)(5) \\ 2(3) + 3(-2) & 2(-7) + 3(5) \end{bmatrix} = \begin{bmatrix} 15-14 & -35+35 \\ 6-6 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{[B]}{\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}} \cdot \underset{[A]}{\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}} = \begin{bmatrix} 15-14 & 21-21 \\ -10+10 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $[A][B] = [B][A] = I$, $[A]$ and $[B]$ are inverses.

b. $$\overset{[A]}{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \overset{[B]}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} 1+0+0 & -2+2+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq I$$

These matrices are <u>not</u> inverses.

In fact, $\det[A] = 0$ ($[A]$ is a singular matrix) so A^{-1} doesn't exist. In general, a matrix with a column or row of all zero entries will be a singular matrix.

8. a. $\left[\begin{array}{cc|cc} 1/2 & 1/3 & 1 & 0 \\ 1/2 & 2 & 0 & 1 \end{array}\right] \xrightarrow[2R_2 \to R_2]{2R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 2 & 0 \\ 1 & 4 & 0 & 2 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 2 & 0 \\ 0 & 10/3 & -2 & 2 \end{array}\right]$

$\xrightarrow{\frac{3}{10}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 2 & 0 \\ 0 & 1 & -3/5 & 3/5 \end{array}\right] \xrightarrow{R_1 - \frac{2}{3}R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 12/5 & -2/5 \\ 0 & 1 & -3/5 & 3/5 \end{array}\right]$

$$\begin{array}{rcccc} R_1 & 1 & \frac{2}{3} & 2 & 0 \\ -\frac{2}{3}R_2 & 0 & -\frac{2}{3} & \frac{2}{5} & \frac{-2}{5} \\ \hline & 1 & 0 & 12/5 & -2/5 \end{array}$$

The system can be written as:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{49}{18} \\ 4/3 \end{bmatrix}$$

$$[A][x] = [b]$$

$$[A]^{-1}[A][x] = [A]^{-1}[b]$$

$$[I][x] = [x] = [A]^{-1}[b] \quad \text{where } [A]^{-1} = \begin{bmatrix} 12/5 & -2/5 \\ -3/5 & 3/5 \end{bmatrix}$$

$$[A]^{-1}[b] = \begin{bmatrix} 12/5 & -2/5 \\ -3/5 & 3/5 \end{bmatrix}\begin{bmatrix} 49/18 \\ 4/3 \end{bmatrix} = \begin{bmatrix} \frac{12}{5}\cdot\frac{49}{18} - \frac{2}{5}\cdot\frac{4}{3} \\ \frac{-3}{5}\cdot\frac{49}{18} + \frac{3}{5}\cdot\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 98/15 - 8/15 \\ -49/30 + 12/15 \end{bmatrix} = \begin{bmatrix} 90/15 \\ \frac{-49}{30} + \frac{24}{30} \end{bmatrix} = \begin{bmatrix} \frac{90}{15} \\ \frac{-25}{30} \end{bmatrix} = \begin{bmatrix} 6 \\ -\frac{5}{6} \end{bmatrix}$$

$$\boxed{x = 6 \quad y = -5/6}$$

b. $\begin{bmatrix} 3 & 2 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -27 \end{bmatrix}$

$[A] \cdot [x] = [b]$

$[A]^{-1}: \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -6 & 1 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow[R_2/5 \to R_2]{R_1/3 \to R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right]$

$\xrightarrow{R_1 - \frac{2}{3}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/15 & -2/15 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right] \quad [A]^{-1} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

$$\begin{array}{rcccc} R_1 & 1 & \frac{2}{3} & \frac{1}{3}\cdot\frac{5}{5} & 0 \\ -\frac{2}{3}R_2 & 0 & -\frac{2}{3} & \frac{-4}{15} & \frac{-2}{15} \\ \hline & 1 & 0 & 1/15 & -2/15 \end{array}$$

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$$[X] = [A]^{-1}[b] = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 6 \\ -27 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{15}(6) - \frac{2}{15}(-27) \\ \frac{2}{5}(6) + \frac{1}{5}(-27) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{15} + \frac{54}{15} \\ \frac{12}{5} - \frac{27}{5} \end{bmatrix} = \begin{bmatrix} \frac{60}{15} \\ \frac{-15}{5} \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\boxed{x = 4,\ y = -3 \text{ or } (x,y) = (4,-3)}$$

8.c. $\begin{aligned} 2x - 3y &= -7 \\ -x - 9y &= 3 \end{aligned} \rightarrow \begin{bmatrix} 2 & -3 \\ -1 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$ $[A][X] = [b]$

For Cramer's rule: $D = \begin{vmatrix} 2 & -3 \\ -1 & -9 \end{vmatrix} = 2(-9) - (-1)(-3) = -18 - 3 = -21$

Replacing the x-coefficients in [A] with the [b] gives:

$$D_x = \begin{vmatrix} -7 & -3 \\ 3 & -9 \end{vmatrix} = (-7)(-9) - (3)(-3) = 63 + 9 = 72$$

then $x = \frac{D_x}{D} = \frac{72}{-21} = -\frac{24}{7}$

Replacing the y-coefficients in [A] with [b] gives:

$$D_y = \begin{vmatrix} 2 & -7 \\ -1 & 3 \end{vmatrix} = 2(3) - (-1)(-7) = 6 - 7 = -1$$

$$y = \frac{D_y}{D} = \frac{-1}{-21} = \frac{1}{21}$$

so $\boxed{(x,y) = \left(\frac{-24}{7}, \frac{1}{21}\right)}$

8.4 $3x-5y=19$
$5x-6z=-9$
$-5y-z=-6$

$$\longrightarrow \begin{bmatrix} 3 & -5 & 0 \\ 5 & 0 & -6 \\ 0 & -5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ -9 \\ 6 \end{bmatrix}$$

$$D = \begin{vmatrix} 3 & -5 & 0 \\ 5 & 0 & -6 \\ 0 & -5 & -1 \end{vmatrix} = 3(0)(-1) + (-5)(-6)(0) + 0(5)(-5) - 0(0)(0) - (-5)(5)(-1) - (3)(-6)(-5)$$

$$= 0+0+0-0-25-90 = -115$$

$$D_x = \begin{vmatrix} 19 & -5 & 0 \\ -9 & 0 & -6 \\ 6 & -5 & -1 \end{vmatrix} = 19(0)(-1) + (-5)(-6)(6) + 0(-9)(-5) - (0)(0)(6) - (-5)(-9)(-1) - 19(-6)(-5)$$

$$= 0+180+0-0+45-570$$

$$= 225-570 = -345$$

$$x = \frac{D_x}{D} = \frac{-345}{-115} = 3$$

$$D_y = \begin{vmatrix} 3 & 19 & 0 \\ 5 & -9 & -6 \\ 0 & 6 & -1 \end{vmatrix} = 3(-9)(-1) + (19)(-6)(0) + (0)(5)(6) - 0(-9)(0) - 19(5)(-1) - 3(-6)(6)$$

$$= 27-0+0-0+95+108$$

$$= 230$$

$$y = \frac{D_y}{D} = \frac{230}{-115} = -2$$

$$D_z = \begin{vmatrix} 3 & -5 & 19 \\ 5 & 0 & -9 \\ 0 & -5 & 6 \end{vmatrix} = 3(0)(6) + (-5)(-9)(0) + (19)(5)(-5) - 19(0)(0) - (-5)(5)(6) - 3(-9)(-5)$$

$$= 0+0-475-0+150-135$$

$$= -610+150 = -460$$

$$z = \frac{D_z}{D} = \frac{-460}{-115} = 4$$

$$\boxed{(x,y,z) = (3,-2,4)}$$

Additional Resources

1. Go to http://www.kutasoftware.com/freeipc.html

2. Under **"Matrices and Systems"** find**:**
    - Matrix operations
    - Matrix inverses and determinants
    - Matrix equations
    - Cramer's Rule
    - Multivariable linear systems and row operations

You can print out the worksheets and work on them. The solutions are provided at the end of the worksheets

3. For help please contact the ***Math Assistance Area*.**